Constraining Newtonian stellar configurations in $f(R)$ theories of gravity

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We consider general metric $f(R)$ theories of gravity by solving the field equations in the presence of a spherical static mass distribution by analytical perturbative means. Expanding the field equations systematically in $\mathcal{O}(G)$, we solve the resulting set of equations and show that $f(R)$ theories which attempt to solve the dark energy problem very generally lead to $\gamma_{PPN} = 1/2$ in the solar system. This excludes a large class of theories as possible explanations of dark energy. We also present the first order correction to γ_{PPN} and show that it cannot have a significant effect.

I. INTRODUCTION

The dark energy problem remains central in modern day cosmology. Since the matter only, homogeneous universe within the framework of general relativity is in conflict with cosmological observations, the assumptions behind this model have been questioned. The most popular modification is to consider a universe filled with other, more exotic forms of matter, the cosmological constant being the leading natural candidate. Other ways to tackle the dark energy problem are then to relax the assumption of homogeneity or modify the theory of gravity.

In recent years, a particular modification of gravity, the $f(R)$ gravity models that replace the Einstein-Hilbert action of general relativity (GR) with an arbitrary function of the curvature scalar (see *e.g.* [1, 2, 3, 4, 5, 6, 7, 8, 9] and references therein) have been extensively studied. Naive modification of the gravitational action is not without challenges, however, and obstacles including cosmological constraints (see *e.g.* [38, 39, 40] and references therein), instabilities [11, 12, 13], solar system constraints (see *e.g.* [14, 15, 16, 21] and references therein) and evolution large scale perturbations [17, 18, 19] need to be overcome. In addition, a number of consistency requirements need to be satisfied (see *e.g.* [20, 22] and references therein).

One of the most direct and strictest constraints on any modification of gravity comes from observations of out nearby space-time *i.e.* the solar system. This is often done by conformally transforming the theory to a scalar-tensor theory and then considering the Parameterized Post-Newtonian (PPN) limit [23, 24] (see also [25, 26] for a discussion). The question of validity of the solar system constraints $f(R)$ theories has been extensively discussed in the literature and not completely without controversy. The opinions on the viability of $f(R)$ theories have been divided from more or less skeptical [27, 28, 29, 30] to approving [31, 32] depending on the point of view of the author.

The essence of the discussion has been the question of validity of the Schwarzschild-de Sitter (SdS) metric as the correct metric in the solar system. The SdS metric is a vacuum solution to a large class of $f(R)$ theories of gravity but due to the higher-derivative nature of metric $f(R)$ theories, it is not unique. Other solutions can also be constructed in empty space, in the presence of matter and in a cosmological setting (see *eg.* [10, 33, 34]).

In light of recent literature [27, 28, 29], the validity of the solar system constraints has become clear and it is now understood that the equivalent scalar-tensor theory results are valid in a particular limit that corresponds to the limit of light effective scalar in the In terms of the $f(R)$ theory, this is equivalent to requiring that one can approximate the trace of the field equations by Laplace's equation [28]. As a result, the often considered $R-\mu^4/R$ theory [1] (the CDTT model) is not consistent with the Solar System constraints in this limit, if the $1/R$ term is to drive late time cosmological acceleration.

In [27] the CDTT model was considered by linearizing around a static de Sitter spacetime and solving the trace equation in terms of $R(r)$, resulting in a spacetime outside the star where $\gamma = 1/2$. This result was then generalized for a general $f(R)$ theory in [28] by studying the space-time outside a spherical mass distribution and expanding $f(R)$ in terms of a perturbation in R. Again solving the trace equation leads to an outside solution with $\gamma = 1/2$ as long as the effective scalar mass is light. A somewhat different approach was followed in [35], where the trace equation was first written in terms $F(r) \equiv df/dR$ in the perturbative expansion. Solving the trace equation then leads to $\gamma = 1/2$ outside the star.

In this paper we follow the latter approach by viewing F along with the metric as independent functions. By expanding all quantities in G and solving the resulting equations inside and outside the star for a general $f(R)$ theory, we find that generally, $\gamma_{PPN} = 1/2 + \mathcal{O}(G)$ outside the star and the scalar curvature is $\mathcal{O}(G^2)$ everywhere. We also idenfity the first order correction to γ_{PPN} and show that it cannot have a significant effect. Only if initial conditions inside the star are fine-tuned such that the scalar curvature follows the matter density like in GR [21, 35] can these bounds be evaded.

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II. $f(R)$ GRAVITY FORMALISM

The action for $f(R)$ gravity is $(c = 1)$

$$S = \int d^4x \sqrt{-g}\Big(\frac{1}{16\pi G} f(R) + \mathcal{L}_m\Big). \quad (1)$$

The field equations resulting in the so-called metric approach are reached by variating with respect to $g_{\mu\nu}$:

$$F(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R) + g_{\mu\nu}\Box F(R) = 8\pi G T^m_{\mu\nu}, \quad (2)$$

where $T^m_{\mu\nu}$ is the standard minimally coupled stress-energy tensor and $F(R) \equiv df/dR$.

Contracting the field equations and assuming that we can describe the stress-energy tensor with a perfect fluid, we get

$$F(R)R - 2f(R) + 3\Box F(R) = 8\pi G(\rho - 3p). \quad (3)$$

In this letter we consider spherically symmetric static fluid configurations and adopt a metric, which reads in spherically symmetric coordinates as

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2. \quad (4)$$

By taking suitable linear combinations of the field equations they can be written in the following form:

$$\frac{F\,A'}{r\,A} + \frac{F\,B'}{r\,B} + \frac{A'\,F'}{2\,A} + \frac{B'\,F'}{2\,B} - F'' \;=\; 8\,G\,\pi\,A\,(\rho + p) \quad (5)$$

$$-\frac{F}{r^2} + \frac{A\,F}{r^2} + \frac{F\,A'}{2\,r\,A} + \frac{F\,B'}{2\,r\,B} - \frac{F\,A'\,B'}{4\,A\,B} - \frac{F\,B'^2}{4\,B^2} - \frac{F'}{r} + \frac{B'\,F'}{2\,B} + \frac{F\,B''}{2\,B} \;=\; 8\,G\,\pi\,A\,(\rho + p) \quad (6)$$

$$A\,(2f(R) - R\,F(R)) + \frac{6\,F'}{r} - \frac{3\,A'\,F'}{2\,A} + \frac{3\,B'\,F'}{2\,B} + 3\,F'' \;=\; -8\,G\,\pi\,A\,(\rho - 3p), \quad (7)$$

where prime indicates a derivation with respect to r, $' \equiv d/dr$ and we have written f and F as functions of the radial coordinate r expect in combination $2f(R) - R\,F(R)$, which we will expand in terms of curvature R.

The corresponding equation of continuity is

$$\frac{p'(r)}{\rho(r) + p(r)} = -\frac{1}{2}\frac{B'(r)}{B(r)}. \quad (8)$$

When pressure is negligible, it is easy to see that B must be a constant. This is, however, not acceptable and therefore an adequate perturbation expansion is needed.

III. PERTURBATIVE EXPANSION AND ITS SOLUTIONS

We expand the metric as well as F with G as an expansion parameter:

$$\begin{aligned} A(r) &= 1 + G\,A_1(r) + \mathcal{O}(G^2), \\ B(r) &= B_0 + G\,B_1(r) + \mathcal{O}(G^2), \\ F(r) &= F_0 + G\,F_1(r) + \mathcal{O}(G^2), \\ p(r) &= p_0 + G\,p_1(r) + \mathcal{O}(G^2). \end{aligned}$$

Note, that we consider the density profile $\rho(r)$ to be a fixed function and also that B_0 and F_0 are constants. From the expansion of A and B, one can also read out an expansion for R:

$$R = R_0 + GR_1 + \mathcal{O}(G^2). \quad (9)$$

From the equation of continuity we see that at $\mathcal{O}(G^0)$ pressure is constant and exactly zero, $p_0 = 0$, simply because it vanishes in empty space. Therefore pressure effects are always $\mathcal{O}(G^2)$ and do not contribute to the $\mathcal{O}(G^1)$ expansion.

The $2f - FR$ term in the third field equations is crucial in determining the behaviour of the solution. In general, for a $f(R)$ dark energy model, this term is negligible and can be omitted, at least in the first order approximation. This is demonstrated explicitly for the CDTT in model and discussed more generally in [35], where it is argued that the non-linear term is completely negligible, barring fine tuning. This argument is easily understandable in a general model since in the vacuum $2f - FR \sim G\rho_{DE} \ll G\rho$ for any stellar matter configuration. Note that this will in general be true also outside a stellar configuration as the dark matter will completely dominate over the cosmological term. Hence, in the trace equation, the non-linear terms can be dropped, unless the initial conditions are fine-tuned. We will return to the fine-tuned condition, or the Palatini limit [21, 35], later.

More formally, the same conclusion can be confirmed by using an expansion in G for the non linear-terms as well: $2f(R) - F(R)R = 2f(R_0) - F(R_0)R_0 + (F(R_0) - F'(R_0)R_0)R_1 + \mathcal{O}(G^2)$ Evidently, the expansion point R_0 has to be such, that it corresponds to the correct background of the theory, *i.e.* $2f(R_0) - F(R_0)R_0 = 0$. Then

expanding up to first order in G, the field equations are

$$\begin{aligned}
\frac{F_0\,A_1'}{r} + \frac{F_0\,B_1'}{B_0\,r} - F_1'' &= 8\,\pi\,\rho, \\
\frac{F_0\,A_1}{r^2} - \frac{F_0\,A_1'}{2\,r} + \frac{F_0\,B_1'}{2\,B_0\,r} - \frac{F_1'}{r} + \frac{F_0\,B_1''}{2\,B_0} &= 8\,\pi\,\rho, \quad (10) \\
I_1 R_1 + \frac{6\,F_1'}{r} + 3\,F_1'' &= -8\,\pi\,\rho,
\end{aligned}$$

where $I_1 = F(R_0) - F'(R_0)R_0$ is a constant and $F_0 = F(R_0)$.

The set of equations (10) can be straightforwardly solved leading to $\mathcal{O}(G)$ functions:

$$F(r) = F_0 - \frac{2}{3}G\int_0^r \frac{m(r)}{r^2}\,dr \quad (11)$$

$$A(r) = 1 + \frac{4G}{3F_0}\frac{m(r)}{r} \quad (12)$$

$$B(r) = B_0\Big(1 + \frac{8G}{3F_0}\int_0^r \frac{m(r)}{r^2}\Big)\,dr, \quad (13)$$

where

$$m(r) \equiv \int_0^r 4\pi r^2 \rho\,dr. \quad (14)$$

Inserting this solution back to expression of the curvature scalar, we find that $R_1 = 0$, *i.e.*, R is $\mathcal{O}(G^2)$. It is crucial that in deriving the solution (11), we have assumed that $A,\ B,\ F$ are regular at the origin.

The PPN-parameter is now straightforwardly calculable:

$$\gamma_{PPN} = \frac{1}{2}\Big(1 - r\frac{m'}{m}\Big) + \frac{2G}{3F_0}\frac{\left(2\int_0^r \frac{m}{r^2}\,dr + m'\right)\,(m - r\,m')}{m}. \quad (15)$$

It is easy to see that, at the boundary of the star $\gamma_{PPN} \to 1/2 + \mathcal{O}(G)$. This behaviour was also observed in numerical studies [35, 36]. From the first order correction one can furthermore conclude that if one wishes corrections to be effective at zeroth order, F_0 needs to be of order G. However, looking at the continuity equation, Eq. (8), we find that

$$-\,r^2 p' = \frac{4}{3}\frac{G}{F_0}\rho\, m(r) + \mathcal{O}(G^2). \quad (16)$$

Comparing this with the Newtonian result, $-r^2p' = 4G\rho m(r)$, we see that if $F_0 \sim \mathcal{O}(G)$, the effective Newton's constant is orders of magnitude larger than the one in Newton's theory (or GR), resulting in stars with a completely different mass to radius relationship than the one observed. Furthermore, from the continuity equation, we can read that unless $F_0 \approx 4/3$ to a high precision, a star with the same density profile, and hence total mass, will have a different radius than in GR. This behaviour was already observed in [36].

In general the results described in this section will apply even when $2f - FR \sim R$, *i.e.* when $f(R) = R + c_1R^2$. Because in the approximation described above, $R \sim \mathcal{O}(G^2)$, it is easy to see that that this term will play no role in the trace equation. Similarly for higher order terms in R. One can avoid the constraint only if F has no G order correction. In this case, the $\Box F$, term is negligible in trace equation and we recover the GR results, or the Palatini limit [21, 35]. Alternatively, if one relaxes the regularity constraint of the metric at the origin, one can also avoid the constraint as demonstrated in [36] for the CDTT model.

A. Recovering the general relativity

In the Palatini limit, where the trace-equation is similar than in the Palatini formalism, the theory is fine-tuned so that $2f - FR \approx R \approx -8\pi G\rho$ throughout (see [35] for a numerical example). This is the mechanism that allows one to construct solutions that are consistent with solar system observations [21]. In the Palatini limit, the field equations read as

$$F(r) \simeq 1 \quad (17)$$

$$A(r) \simeq 1 + \frac{2Gm(r)}{r} \quad (18)$$

$$B(r) \simeq B_0\Big(1 + 2G\int_0^r \frac{m(r)}{r^2}\Big)\,dr. \quad (19)$$

The γ_{PPN} parameter is easily calculable:

$$\gamma_{PPN} \simeq 1 - r\frac{m'}{m} + \mathcal{O}(G). \quad (20)$$

Therefore, in this limit $\gamma_{PPN} \to 1$ at the surface of the star. However, as shown in [35] for the CDTT model, this limit can be unstable time leading to the Dolgov-Kawasaki instability [11]. Stable theories are considered in [21] and studied analytically in [37].

IV. DISCUSSION AND CONCLUSIONS

In this letter we have considered a general metric $f(R)$ theory in the presence of matter by analyzing the field equations by perturbative means in linear order in the Newton's constant G. We have shown explicitly that for a typical star, any modification of gravity from GR will naturally lead to physically unacceptable value $\gamma_{PPN} = 1/2$. This places a very strong constraint on any $f(R)$ theory, in particular when acting as a dark energy candidate. Furthermore, even if the gravity theory is not motivated by cosmology, but by other arguments, such as quantum gravity, the presence of non-linear terms can still lead to a space-time inconsistent with observations.

In this order of perturbation theory we can recover the observationally acceptable space-time, only when $F = df/dR$ has no order G correction. Such a constraint indicates fine-tuning in the initial values of the solution so that one remains in the high curvature limit, $R \sim G\rho$

throughout. However, the stability of such a fine-tuned solution may be problematic [35], although possible to obtain [21, 37, 38].

Since our analysis is of order $\mathcal{O}(G^1)$, further study on the system, in particular second order perturbations in G, may affect the conclusions. Indeed, our analysis shows that the first order perturbation theory is essentially independent on the details of the underlying $f(R)$ theory. The only piece of information used was the knowledge that there are higher order derivatives in the equations of motion, *i.e.* that the theory is not GR. New effects may appear in higher order perturbation theory, where finally the dependence on the functional form of $f(R)$ should become evident. However, our results suggest that unless the solution is fine-tuned so that $R \sim G\rho$ throughout the mass distribution, a naive modification where a small correction is added to the Einstein-Hilbert action to solve the dark energy problem is not likely to pass the solar system constraints.

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